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Identity of the SU(1, 1) and SU(2) Clebsch–Gordan coefficients coupling unitary discrete representations

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Abstract. The Clebsch–Gordan (CG) coefficients coupling two positive discrete series or two negative discrete series of irreducible unitary representations (IUR) of SU(1, 1) are shown to be identical to SU(2) CG coefficients. The transformations changing the indices of the SU(1, 1) CG coefficients into those of the identical SU(2) CG coefficients are derived and shown to transform the SU(2) CG coefficients into SU (1, 1) CG coefficients. The associated phases are discussed. General index transformations are derived and used to generate SU(2) CG coefficient symmetries, among which are some of the more abstract Regge symmetries. A simple invariance property of the intermediate SU(1, 1) CG coefficients is at the base of our symmetries. Our demonstrations are carried out with coexistent IUR of the tensor product groups SU(1, 1)⊕SU(1, 1) and SU(2)⊕SU(2) embedded in a simple encompassing group structure, the most degenerate discrete IUR of SU(2, 2).

1. Introduction

The purpose of this paper is to prove that the SU(1, 1) Clebsch–Gordan (CG) coefficients coupling two positive or two negative discrete series of irreducible unitary representations (IUR) are identical to SU(2) CG coefficients. This result can be exploited for finding numerical values of these discrete series SU(1, 1) CG coefficients by transforming their indices with (22) into SU(2) indices and using the extensive tables of SU(2) CG coefficients (Rotenberg *et al* 1959). As a by-product in the derivation of the general index transformations, we are able to shed some light on the abstract SU(2) CG coefficient symmetries of Regge (1958).

Starting with Bargmann's (1947) article, the non-compact group SO(2, 1) and its covering group SU(1, 1) have been studied extensively in the mathematical physics literature. In particular, the CG coefficients coupling two discrete non-unitary, ie finite-dimensional, irreducible representations (IR) have been shown to be identical to SU(2) CG coefficients (Bargmann 1947, Holman and Biedenharn 1966). Since the SU(1, 1) and SU(2) Lie algebras have the same complex extension, this result is easily proved by placing the basis states of the non-unitary, finite-dimensional SU(1, 1) IR in one-to-one correspondence with the states of SU(2) IUR of the same dimensions. Such a direct match with SU(2) IUR basis states clearly is impossible when considering the CG coefficients coupling discrete unitary, ie infinite-dimensional, IUR of SU(1, 1). Thus, up to now, the recognized connection between the SU(1, 1) and SU(2) CG coefficients coupling discrete unitary IR is the analytic continuation in the representation parameters (Holman and Biedenharn 1966, Wang 1970).

By stepping away from the consideration of individual SU(1, 1) and SU(2) IUR to a study of the IUR of the tensor product groups SU(1, 1)⊕SU(1, 1) and SU(2)⊕SU(2) defined on one general set of basis states, we can match the limited sets of SU(1, 1) tensor product states appearing in the expansion of a discrete SU(1, 1) IUR state with a corresponding set of SU(2) tensor product states appearing in the expansion of an SU(2) IUR state. The important point is that the two expansions, and thus the SU(1, 1) and SU(2) CG coefficients, will be identical even if only the limited sets of states involved are identical. The most degenerate, discrete IUR of SU(2, 2) (Yao 1967, Barut and Böhm 1970) provide a natural structure in which this can be done. These SU(2, 2) IUR have a physical significance as the generalized hydrogen atom or dyonium dynamical group IUR (Barut and Bornzin 1971, Barut *et al* 1974). In this physical picture, the SU(1, 1) and SU(2) CG coefficients turn out to be the basis transformation coefficients relating 'parabolic' to 'spherical' basis states, and the index transformations (22) are then just the notational transformations of these SU(2, 2) states from an SU(1, 1) representation parameter description to an SU(2) representation parameter description.

In the construction of our proofs, only the positive discrete series IUR of SU(1, 1) will be used; a brief discussion of the negative discrete series IUR is included in § 7. In § 2, the encompassing group structure of the most degenerate discrete IUR of SU(2, 2) is sketched. The SU(2)⊕SU(2) and SU(1, 1)⊕SU(1, 1) IUR embedded in this structure are presented next, in §§ 3 and 4 respectively. In § 5.1, the CG coefficients and the index transformation, along with a brief discussion of phases, are presented. In § 5.2, the general index transformations are derived. These lead, as discussed in § 6, to SU(2) CG symmetries, some of which are Regge symmetries.

2. The group SU(2, 2)

The group SU(2, 2) (Yao 1967) is locally isomorphic to the group O(4, 2), and can be generated by the Lie algebra of antisymmetric tensors $J_{\alpha\beta}$, where α and β take values from the set $(1, \ldots, 6)$. $J_{\alpha\beta}$ satisfy the commutation rules

$$[J_{\alpha\beta}, J_{\gamma\beta}] = -\mathrm{i}g_{\beta\beta}J_{\alpha\gamma} \tag{1}$$

where $g_{\alpha\beta}$ is a diagonal metric tensor with the elements $g_{11} = g_{22} = g_{33} = g_{44} = -1$ and $g_{55} = g_{66} = +1$. The oscillator or boson realization of the most degenerate discrete IUR of SU(2, 2) can be constructed with the four pairs $(a_1^\dagger, a_1)$, $(a_2^\dagger, a_2)$, $(b_1^\dagger, b_1)$ and $(b_2^\dagger, b_2)$ of boson creation and annihilation operators satisfying

$$[a_r^\dagger, a_s] = [b_r^\dagger, b_s] = \delta_{rs}.$$

If we define the usual Pauli matrices and a matrix C by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then the generators $J_{\alpha\beta}$ can be written in a two-dimensional spinor notation as

$$\begin{aligned}
J_{ij} &= \tfrac{1}{2}\epsilon_{ijk}(a^\dagger\sigma_k a+b^\dagger\sigma_k b)\\
J_{k4} &= -\tfrac{1}{2}(a^\dagger\sigma_k a-b^\dagger\sigma_k b)\\
J_{i5} &= -\tfrac{1}{2}(a^\dagger\sigma_i Cb^\dagger-aC\sigma_i b)\\
J_{i6} &= -\tfrac{1}{2}\mathrm{i}(a^\dagger\sigma_i Cb^\dagger+aC\sigma_i b)\\
J_{45} &= -\tfrac{1}{2}\mathrm{i}(a^\dagger Cb^\dagger-aCb)\\
J_{46} &= \tfrac{1}{2}(a^\dagger Cb^\dagger+aCb)\\
J_{56} &= \tfrac{1}{2}(a^\dagger a+b^\dagger b+2)
\end{aligned}\tag{2}$$

where i and j take the values 1, 2, and 3. The most degenerate discrete (dyonium) IUR of SU(2, 2) are uniquely labelled by the eigenvalues μ of the operator S:

$$S \equiv \tfrac{1}{2}(a^\dagger a-b^\dagger b) = \tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2-b_1^\dagger b_1-b_2^\dagger b_2). \tag{3}$$

S commutes with all generators $J_{\alpha\beta}$ of the SU(2, 2) Lie algebra (2). The orthonormal 'parabolic' basis states are the eigenstates of the operator S together with the three commuting generators J_{12}, J_{56}, J_{34} of the Cartan subalgebra. The diagonalization of $S, J_{12}, J_{56}, J_{34}$ is, as can easily be seen by taking their linear combinations, equivalent to the diagonalization of the four number operators $a_1^\dagger a_1$, $a_2^\dagger a_2$, $b_1^\dagger b_1$, and $b_2^\dagger b_2$. Thus, orthogonal 'parabolic' basis states are simply products of monomials in $a_1^\dagger$, $a_2^\dagger$, $b_1^\dagger$, and $b_2^\dagger$. When defining the general index transformations, we shall use the number operators. We write the 'parabolic' basis states as $|\mu, m, n, \alpha\rangle$, where the labels (μ, m, n, α) are the eigenvalues, in the same order, of the set of operators $(S, J_{12}, J_{56}, J_{34})$. The 'spherical' basis states will be defined later, in conjunction with the CG coefficients.

3. The tensor product group SU(2)⊕SU(2)

The SU(2, 2) tensor product subgroup SU(2)⊕SU(2), which is locally isomorphic to O(4), is generated by the subalgebra $J_{\alpha\beta}$, where (α, β) take their values from the set (1, 2, 3, 4). From (2), the generators of the two SU(2) groups composing the SU(2)⊕SU(2) are easily seen to be

$$(J_1)_{ij} = \tfrac{1}{2}\epsilon_{ijk}(a^\dagger\sigma_k a), \qquad (J_2)_{ij} = \tfrac{1}{2}\epsilon_{ijk}(b^\dagger\sigma_k b). \tag{4}$$

Here i, j, and k take values from 1 to 3. With $(J)_{ij} = \epsilon_{ijk}(J)_k$, the SU(2) Casimir invariants $Q \equiv J^2$ are defined by

$$Q \equiv (J)^2 = \tfrac{1}{2}\sum g^{ij}g^{kl}J_{ik}J_{jl} = (J)_1^2+(J)_2^2+(J)_3^2. \tag{5}$$

An IUR of SU(2)⊕SU(2) can be constructed by taking the tensor product of a spin j_1 with a spin j_2 SU(2) IUR; a complete set of SU(2)⊕SU(2) IUR basis states are then:

$$|j_1m_1j_2m_2\rangle = |j_1m_1\rangle|j_2m_2\rangle = N_{j_1m_1}a_1^{\dagger j_1+m_1}a_2^{\dagger j_1-m_1}N_{j_2m_2}b_1^{\dagger j_2+m_2}b_2^{\dagger j_2-m_2}|0\rangle \tag{6}$$

with $N_{jm}^{-2} = [(j+m)!(j-m)!]$. The action of the raising and lowering operators

$(J_r)^\pm = (J_r)_1 \pm i(J_r)_2$, of $(J_r)_3$, and of the Casimirs on (6) are

$$\begin{aligned}(J_1)^\pm|j_1m_1j_2m_2\rangle &= [(j_1\mp m_1)(j_1\pm m_1+1)]^{1/2}|j_1,m_1\pm 1,j_2m_2\rangle\\ (J_2)^\pm|j_1m_1j_2m_2\rangle &= [(j_2\mp m_2)(j_2\pm m_2+1)]^{1/2}|j_1m_1,j_2,m_2\pm 1\rangle\\ (J_r)_3|j_1m_1j_2m_2\rangle &= m_r|j_1m_1j_2m_2\rangle\\ (J_r)^2|j_1m_1j_2m_2\rangle &= j_r(j_r+1)|j_1m_1j_2m_2\rangle.\end{aligned} \tag{7}$$

Since the states $|j_1m_1j_2m_2\rangle$ are simply products of monomials of the four boson raising operators, they are identical to the SU(2, 2) 'parabolic' basis states $|\mu, m, n, \alpha\rangle$, if the operators of the set $(S, J_{12}, J_{56}, J_{34})$ have matching eigenvalues on both states:

$$\begin{aligned}\mu &= j_1-j_2\\ m &= m_1+m_2\\ n &= j_1+j_2+1\\ \alpha &= m_2-m_1.\end{aligned} \tag{8}$$

These relations define the index transformation from the SU(2, 2) 'parabolic' state notation to the SU(2) tensor product notation. All SU(2)⊕SU(2) IUR satisfying the condition $\mu = j_1-j_2$ are contained once in one SU(2, 2) spin μ IUR.

The alternate SU(2)⊕SU(2) IUR basis states corresponding to the 'spherical' basis states are defined by the canonical reduction chain

$$\mathrm{SU}(2)\oplus\mathrm{SU}(2) \supset \mathrm{SU}(2) \supset \mathrm{SU}(1). \tag{9}$$

The SU(2) CG coefficients will satisfy the Condon–Shortley phase convention (Edmonds 1957) if the SU(2) subgroup generated by (J_{12}, J_{31}, J_{23}) and the SU(1) subgroup generated by J_{12} are chosen for this chain. The generators of the SU(2) reduction chain group are

$$\begin{aligned}J_{23} &= J_1 = (J_1)_1+(J_2)_1\\ J_{13} &= J_2 = (J_1)_2+(J_2)_2\\ J_{12} &= J_3 = (J_1)_3+(J_2)_3.\end{aligned} \tag{10}$$

The new states $|(j_1j_2)jm\rangle$, transforming as IUR states of SU(2), are related to the tensor product states (6) by the unitary transformation defining SU(2) CG coefficients:

$$|(j_1j_2)jm\rangle = \sum_{m_1m_2} \langle j_1m_1j_2m_2|(j_1j_2)jm\rangle|j_1m_1j_2m_2\rangle\,\delta_{m_1+m_2,m}. \tag{11}$$

The eigenvalues $j(j+1)$ of the Casimir operator $Q = J^2 = J_1^2+J_2^2+J_3^2$ are constrained to the usual range $|j_1-j_2| \leqslant j \leqslant j_1+j_2$. The set of operators (S, J_{12}, J_{56}, J^2), where J^2 has taken the place of the generator J_{34}, forms an alternate complete set of commuting SU(2, 2) operators. The eigenstates are defined to be the 'spherical' basis states $|\mu, m, n; j(j+1)\rangle$. The expansion (11) of $|(j_1j_2)jm\rangle$ in terms of the states $|j_1m_1j_2m_2\rangle$ can then be written as the expansion of the 'spherical' states in terms of the 'parabolic' states with the help of the index transformation (8):

$$|\mu, m, n; j(j+1)\rangle = \sum_\alpha \langle \mu, m, n, \alpha|\mu, m, n; j(j+1)\rangle|\mu, m, n, \alpha\rangle. \tag{12}$$

4. The tensor product group SU(1, 1)⊕SU(1, 1)

The SU(2, 2) tensor product subgroup SU(1, 1)⊕SU(1, 1), which is locally isomorphic to O(2, 2), can be generated by the subalgebra $J_{\alpha\beta}$, where (α, β) take their values from the set (3, 4, 5, 6). A study of the SU(1, 1)⊕SU(1, 1) Lie algebra reveals that the two SU(1, 1) subgroups composing the tensor product group are generated by

$$\begin{aligned} (N_1)_1 &= \tfrac{1}{2}(b_1^\dagger a_2^\dagger + b_1 a_2); & (N_2)_1 &= \tfrac{1}{2}(a_1^\dagger b_2^\dagger + a_1 b_2) \\ (N_1)_2 &= -\tfrac{1}{2}\mathrm{i}(b_1^\dagger a_2^\dagger - b_1 a_2), & (N_2)_2 &= -\tfrac{1}{2}\mathrm{i}(a_1^\dagger b_2^\dagger - a_1 b_2) \\ (N_1)_3 &= \tfrac{1}{2}(b_1^\dagger b_1 + a_2^\dagger a_2 + 1); & (N_2)_3 &= \tfrac{1}{2}(a_1^\dagger a_1 + b_2^\dagger b_2 + 1). \end{aligned} \tag{13}$$

With the identifications $(N_r)_{ij} = \epsilon_{ijk}(N_r)_k$, the Lie algebras (13) satisfy the usual SU(1, 1) commutation rules (Bargmann 1947)

$$[(N_r)_{ij}, (N_s)_{ik}] = -\mathrm{i}\delta_{rs} g'_{ii} (N_r)_{jk}.$$

The diagonal metric tensor g'_{ij} takes the values $g'_{11} = g'_{22} = -1$ and $g'_{33} = +1$, reflecting the local isomorphism of SU(1, 1) with SO(2, 1). The SU(1, 1) Casimir operators $Q \equiv K^2$ are defined by

$$Q \equiv (K)^2 = \tfrac{1}{2}\sum g'^{ij} g'^{lk} (N)_{il}(N)_{jk} = (N)_3^2 - (N)_1^2 - (N)_2^2. \tag{14}$$

The basis states of an SU(1, 1)⊕SU(1, 1) IUR can be constructed by taking the tensor product of a k_1 with a k_2 positive discrete series IUR:

$$|k_1 n_1 k_2 n_2\rangle = |k_1 n_1\rangle |k_2 n_2\rangle = N_{k_1 n_1} b_1^{\dagger n_1 + k_1 - 1} a_2^{\dagger n_1 - k_1} N_{k_2 n_2} a_1^{\dagger n_2 + k_2 - 1} b_2^{\dagger n_2 - k_2} |0\rangle \tag{15}$$

with $N_{kn}^{-2} = [(n+k-1)!(n-k)!]$. The action of the generators $(N)^{\pm} = (N)_1 \pm \mathrm{i}(N)_2$ and $(N)_3$, and of the Casimirs on (15) are

$$\begin{aligned} (N_1)^{\pm}|k_1 n_1 k_2 n_2\rangle &= [(n_1 \pm k_1)(n_1 \mp k_1 \pm 1)^{1/2} |k_1, n_1 \pm 1, k_2 n_2\rangle \\ (N_2)^{\pm}|k_1 n_1 k_2 n_2\rangle &= [(n_2 \pm k_2)(n_2 \mp k_2 \pm 1)]^{1/2} |k_1 n_1, k_2, n_2 \pm 1\rangle \\ (N_r)_3 |k_1 n_1 k_2 n_2\rangle &= n_r |k_1 n_1 k_2 n_2\rangle \\ K_r^2 |k_1 n_1 k_2 n_2\rangle &= k_r(k_r - 1)|k_1 n_1 k_2 n_2\rangle. \end{aligned} \tag{16}$$

The representation labels $2k_r$ are nonzero positive integers, and the state labels n_r can take all values from k_r to plus infinity in positive integer steps.

Since the states (15) are also simply products of monomials of the boson raising operators, they are identical to the SU(2, 2) 'parabolic' states $|\mu, m, n, \alpha\rangle$ whenever the set of operators $(S, J_{12}, J_{56}, J_{34})$ has matching eigenvalues on both states:

$$\begin{aligned} \mu' &= k_2 - k_1 \\ m' &= k_1 + k_2 - 1 \\ n' &= n_1 + n_2 \\ \alpha' &= n_1 - n_2. \end{aligned} \tag{17}$$

These relations represent the index transformations from an SU(2, 2) 'parabolic' state notation to the SU(1, 1) tensor product notation.

The alternate set of SU(1, 1)⊕SU(1, 1) states, ie states corresponding to 'spherical' basis states, are defined by the canonical reduction chain

$$\mathrm{SU}(1,1)\oplus\mathrm{SU}(1,1) \supset \mathrm{SU}(1,1) \supset \mathrm{SU}(1). \tag{18}$$

If this chain is to define the SU(2, 2) 'spherical' basis states (12), then the SU(1, 1) in the chain must have its Casimir K^2 equal to the SU(2) Casimir J^2 (associated with (11)), and must contain the SU(1) subgroup generated by J_{56}. The special SU(1, 1) subgroup satisfying these demands is the so called 'transition' subgroup (eg Kleinert 1968), generated by J_{46}, J_{45} and J_{56}. Its generators are, in terms of (13),

$$\begin{aligned} J_{46} &= N_1 = -(N_1)_1+(N_2)_1 \\ J_{45} &= N_2 = -(N_1)_2+(N_2)_2 \\ J_{56} &= N_3 = +(N_1)_3+(N_2)_3. \end{aligned} \tag{19}$$

The minus signs associated with the (N_1) algebra reflect the special choice of SU(1, 1) necessary, so that $K^2 = J^2$. The new states $|(k_1k_2)kn\rangle$, with K^2 and $J_{56} = N_3$ diagonal and transforming as positive discrete series SU(1, 1) IUR states, are related to the states (15) by the unitary transformation defining SU(1, 1) CG coefficients:

$$|(k_1k_2)kn\rangle = \sum_{n_1n_2} \langle k_1n_1k_2n_2|(k_1k_2)kn\rangle |k_1n_1k_2n_2\rangle\, \delta_{n_1+n_2,n}. \tag{20}$$

The eigenvalues $k(k-1)$ of K^2 are $k = k_1+k_2+I$, where I is zero or a positive integer. The SU(2, 2) operators S, J_{12}, and J_{56}, along with the operator K^2, are diagonal on (20). Since $J^2 = K^2$, the same set of operators which define the 'spherical' states (12) is diagonal on (20); thus, (20) are actually SU(2, 2) 'spherical' basis states $|\mu', m', n'; k(k-1)\rangle$. With the help of the index transformation (17), we can write (20) in an SU(2, 2) notation as

$$|\mu', m', n'; k(k-1)\rangle = \sum_{\alpha'} \langle \mu', m', n', \alpha'|\mu', m', n'; k(k-1)\rangle |\mu', m', n', \alpha'\rangle. \tag{21}$$

5. The CG coefficients and the general index transformations

5.1. The CG coefficients

The reason for writing the SU(2)⊕SU(2) IUR basis transformations (11) and the SU(1, 1)⊕SU(1, 1) IUR basis transformations (20) as basis transformations from the 'parabolic' to the 'spherical' states in an encompassing SU(2, 2) IUR ((12) and (21)) should be clear. The SU(2) CG coefficients in (11) and (12)

$$\langle j_1m_1j_2m_2|(j_1j_2)jm\rangle = \langle \mu, m, n, \alpha|\mu, m, n; j(j+1)\rangle$$

and the SU(1, 1) CG coefficients in (20) and (21)

$$\langle k_1n_1k_2n_2|(k_1k_2)kn\rangle = \langle \mu', m', n', \alpha'|\mu', m', n'; k(k-1)\rangle$$

are identical if the SU(2, 2) states are identical, ie if the SU(2, 2) labels $\mu = \mu'$, $m = m'$, $n = n'$, $\alpha = \alpha'$, and $j = k-1$ are. Since any IUR of the tensor product group SU(1, 1)⊕SU(1, 1) formed from two arbitrary positive discrete SU(1, 1) IUR can be accommodated in one of the most degenerate discrete IUR of SU(2, 2), we have proved in general, that the CG coefficients coupling any two positive discrete IUR of SU(1, 1) are identical to SU(2) CG coefficients. The index transformations relating the SU(1, 1) CG coefficients' indices to the SU(2) CG coefficients' indices are formed from the two individual index transformations (8) and (17) by setting the corresponding SU(2, 2) parameters equal. Thus, the CG coefficients

$$\langle k_1 n_1 k_2 n_2 | (k_1 k_2) k n \rangle = \langle j_1 m_1 j_2 m_2 | (j_1 j_2) j m \rangle$$

if the SU(2) indices are

$$\begin{aligned} j_1 &= \tfrac{1}{2}(n_1 + n_2 + k_2 - k_1 - 1) \\ j_2 &= \tfrac{1}{2}(n_1 + n_2 + k_1 - k_2 - 1) \\ m_1 &= \tfrac{1}{2}(n_2 - n_1 + k_2 + k_1 - 1) \\ m_2 &= \tfrac{1}{2}(n_1 - n_2 + k_2 + k_1 - 1) \\ j &= k - 1 \\ m &= k_1 + k_2 - 1. \end{aligned} \tag{22}$$

By transforming the indices of an SU(2) CG coefficient (Edmonds 1957, p 45, equation (3.6.11)) with (22), the SU(1, 1) CG coefficients can be derived. We find

$$\begin{aligned} &\langle k_1 n_1 k_2 n_2 | (k_1 k_2) k n \rangle \\ &\quad = (-1)^{n_1 - k_1} [(2k-1)(k-k_1-k_2)!(k+k_1-k_2-1)!(k+k_2-k_1-1)! \\ &\qquad \times (k+k_1+k_2-2)!(n-k)!(n_1-k_1)!(n_1+k_1-1)!(n_2-k_2)! \\ &\qquad \times (n_2+k_2-1)!/(n+k-1)!]^{1/2} \sum_\beta (-1)^\beta [\beta!(k-k_1-k_2-\beta)! \\ &\qquad \times (n_1-k_1-\beta)!(n-k-n_1+k_1+\beta)!(2k_1+\beta-1)!(k-k_1+k_2-\beta-1)!]^{-1}. \end{aligned} \tag{23}$$

Apart from the phase factor $(-1)^{n_1 - k_1}$, our CG coefficient agrees with the result of Sannikov (1967). The phase factor has its origin in the particular choice of SU(1, 1) canonical chain reduction subgroup. The minus signs associated with $(N_1)_1$ and $(N_1)_2$ in the SU(1, 1) generators (19) are responsible, as we have checked by carrying through Sannikov's calculation with our phase choice. These minus signs, and thus our phase factor, can be removed by picking an alternate pair of SU(1, 1) and SU(2) reduction chain subgroups. For example, the SU(1, 1) subgroup generated by (J_{56}, J_{35}, J_{36}) has the desired plus signs in its generators. However, its complementary SU(2) subgroup, a subgroup with $J^2 = K^2$ and $J_3 = J_{12}$, is the SU(2) generated by (J_{12}, J_{14}, J_{24}), which has minus signs associated with its $(J_1)_1$ and $(J_1)_2$ (see (10)). The SU(2) generators, thus, do not satisfy the Condon–Shortley phase conventions (Edmonds 1957); the SU(2) CG coefficients, corresponding to the SU(1, 1) CG coefficients without a phase factor, have the extra phase factor $(-1)^{j_1 - m_1}$. The algebraic framework of the SU(2, 2) IUR, thus, forces this complementary phase behaviour on the CG coefficients.

5.2. The general index transformations

Our discussion of the index transformations has been concerned thus far strictly with transforming from an arbitrary SU(1, 1) CG coefficient to an SU(2) CG coefficient. The reverse, from an arbitrary SU(2) CG coefficient to an SU(1, 1) CG coefficient needs more care, as different domains of the SU(2) indices need different index transformations. We can see this most simply by directly comparing the powers of the monomials of the boson creation operators in the SU(1, 1)⊕SU(1, 1) states (15) with those in the SU(2)⊕SU(2) states (6). The representation defining conditions $n_r \geqslant k_r \geqslant \frac{1}{2}$ for the SU(1, 1) IUR, restricts the domains of the SU(2) IUR parameters for which the states (15) and (6) can be identical. For identical states, the inequalities imply that the exponents

$$\begin{aligned} n_1+k_1-1 &= j_2+m_2 \geqslant j_1-m_1 = n_1-k_1 \\ n_2+k_2-1 &= j_1+m_1 \geqslant j_2-m_2 = n_2-k_2. \end{aligned} \tag{24}$$

Defining $\mu = j_1 - j_2$ and $m = m_1 + m_2$, these inequalities can be written as

$$m \equiv m_1+m_2 \geqslant |\mu| \geqslant 0.$$

This is the SU(2) parameter domain for which the index transformation (22) is valid. For other SU(2) representation parameter domains differently constructed SU(1, 1) tensor product states are needed, so that different exponent associations than in (24) must be made. The SU(1, 1) Lie algebras (13) are invariant under the interchange of $(b_1^\dagger, b_1)$ with $(a_2^\dagger, a_2)$, and of $(a_1^\dagger, a_1)$ with $(b_2^\dagger, b_2)$; consequently, the exponents of the boson creation operators $b_1^\dagger$ and $a_2^\dagger$, and of $a_1^\dagger$ and $b_2^\dagger$, can be interchanged in the SU(1, 1) tensor product states (15) without changing the Lie algebra definitions (16). With the possibility of permuting the exponents in (15), SU(1, 1) tensor product states can be constructed for all SU(2) parameter domains. We find four inequivalent SU(2) parameter domains:

$$m \geqslant |\mu|, \qquad \mu = |\mu| \geqslant m \geqslant -|\mu|, \qquad -\mu = |\mu| \geqslant m \geqslant -|\mu|, \qquad m \leqslant -|\mu|. \tag{25}$$

The index transformations can, as is illustrated by (24), be directly read off from the exponents. In the sequence of parameter domains adopted in (25), the general index transformations are

$$\begin{aligned} j_1+m_1 &= (n_2+k_2-1, n_2+k_2-1, n_2-k_2, n_2-k_2) \\ j_1-m_1 &= (n_1-k_1, n_1+k_1-1, n_1-k_1, n_1+k_1-1) \\ j_2+m_2 &= (n_1+k_1-1, n_1-k_1, n_1+k_1-1, n_1-k_1) \\ j_2-m_2 &= (n_2-k_2, n_2-k_2, n_2+k_2-1, n_2+k_2-1) \\ j &= k-1. \end{aligned} \tag{26}$$

The first column corresponds to the domain $m \geqslant |\mu|$, and is the index transformation (22). All of these index transformations lead to the same SU(1, 1) CG coefficient, since the SU(1, 1) Lie algebras are invariant under the permutations.

6. The CG coefficient symmetries

By branching from one SU(1, 1) CG coefficient with the four index transformations (26) to four identical, but differently labelled, SU(2) CG coefficients, we are led to six SU(2) CG symmetry transformations. The two symmetry transformations from the SU(2) parameter domains $m \geqslant |\mu|$ to $-|\mu| \geqslant m$, and from $\mu = |\mu| \geqslant m \geqslant -|\mu|$ to $-\mu = |\mu| \geqslant m \geqslant -|\mu|$, are equivalent to the reversal of the coupling order of the spins j_1 and j_2 and the transformation of m_1 and m_2 into $-m_1$ and $-m_2$ in the SU(2) CG coefficient. This is one of the standard symmetries (eg Edmonds 1957). The four symmetry transformations from the SU(2) parameter domains $|m| \geqslant |\mu|$ to the domains $|\mu| \geqslant m \geqslant -|\mu|$, however, are not any of the standard symmetries, but are among the abstract algebraic symmetries discovered by Regge (1958). On writing the SU(2) CG coefficients in terms of the Regge square symbol, these transformations are seen to be equivalent to the symmetry transformation corresponding to the interchange of the symbol's rows with its columns, ie symmetry c of Regge's original paper.

In terms of the boson realizations, the origin of these symmetries lies in the invariance of the SU(1, 1) Lie algebra relations (16) under the permutations of the exponents of the boson raising operators in each SU(1, 1) IUR of the tensor product state (15). From a model-independent viewpoint, these permutational invariances are equivalent to the invariance of the Lie algebra relations (16) when the Casimir invariant k_r is replaced by $1-k_r$; the SU(1, 1) CG coefficients are then, of course, also invariant with respect to this replacement. We can thus say, that our SU(2) CG coefficient symmetries, including symmetry c of Regge's paper (1958), are a consequence of the invariance of the interrelating SU(1, 1) CG coefficients under the replacements of k_r by $1-k_r$.

7. The negative discrete series

Our discussion has dealt exclusively with the positive discrete series IUR of SU(1, 1). Our algebraic frame is, however, equally well suited for treating the negative discrete series IUR. The SU(2, 2) Lie algebra (2) can be converted into one appropriate for the negative discrete IUR by simply multiplying the boson definitions of the SU(2, 2) generators $J_{\alpha 5}$ in (2) with -1. This operation leaves the SU(2)$\oplus$SU(2) subalgebra invariant and affects only the SU(1, 1)$\oplus$SU(1, 1) subalgebra. The SU(1, 1) subalgebras (13) composing the tensor product have their boson definitions of $(N_r)_2$ and of $(N_r)_3$ changed into their negative. The effect of this replacement is that the eigenvalues of $(N_r)_3$ on a boson creation operator basis state are negative, and that the boson definitions of the raising and lowering operators $(N_r)^{\pm}$ are, correspondingly, interchanged. The subalgebras $(N_r)_i$ have become those for negative discrete IUR. Furthermore, the Casimirs are unchanged, so that the positive discrete series tensor product states $|k_1n_1k_2n_2\rangle$ (15) are now the negative discrete series tensor product states $|k_1, -n_1, k_2, -n_2\rangle$, as labelled by (16). Since the Casimir of the spherical basis states is unchanged, ie $K^2 = J^2$, the boson definitions of the positive discrete series IUR states $|(k_1k_2)kn\rangle$ and the negative discrete series states $|(k_1k_2)k-n\rangle$ are unchanged. Except for notation, the positive discrete states and the negative discrete states are identical, so that the SU(1, 1) CG coefficients defined by (20) are the negative discrete CG coefficients also, and thus identical to SU(2) CG coefficients.

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